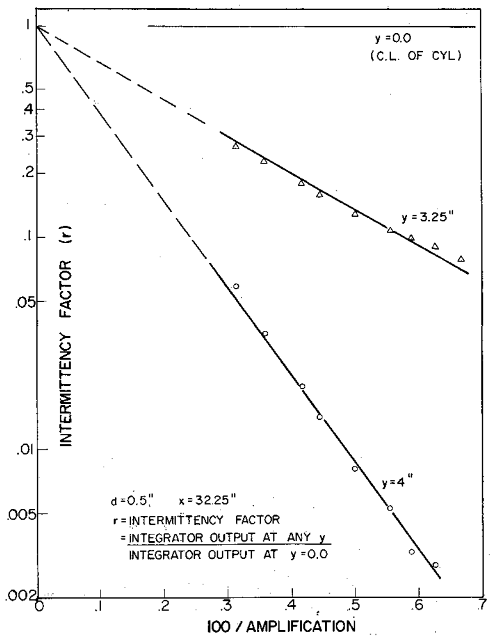

Fig. 3 Effect of amplification on measurement of intermittency factor (normalized integrator output).

by dividing it by the integrator output along the wake axis. This has also been plotted on semilog coordinates. It shows that the intermittency factor may be written as

$$\gamma(y) = \exp(m/G) \tag{2}$$

This seems to arise from the fact that the effect of amplification on measured intermittency is dependent on the intensity of the turbulent bursts. With the same amplification or gain, the percentage increment of γ is lower for high-intensity turbulence than for low-intensity turbulence. This would explain the reduction in slope of the lines in Figs. 2 and 3 as one moves towards $y = 0$.

Regardless of the reasons for the variation of γ with amplification, it remains a fact that the calibration of such circuits is dependent on freestream turbulence level. In the present measurements the circuit was finally calibrated by photographing the rectified, amplified HW signal and measuring the duration of turbulent bursts. The difficulty with such a method is that it becomes progressively more difficult to distinguish between turbulent and nonturbulent regions in the area where $\gamma \approx 0.5$. This is illustrated by the present measurements in the wake of a cylinder of 0.5 in. diam at a distance 32.25 in. downstream. The HW signals were photographed at two points in the y direction. At $y/b = 1.21$, (where b is the nominal wake width given by Prandtl's mixing length theory) $\gamma = 0.102 + 2\%$. At $y/b = 0.95$, $\gamma = 0.487 \pm 12\%$. It is seen that the uncertainty in the calibration is quite high around $\gamma \approx 0.5$. Beyond $\gamma \approx 0.9$ the uncertainty again reduces.

It was attempted to correct the measurements by subtracting the integrator output when the probe was in the freestream from the measurements in the wake. This proved unsuccessful. Further details of the measurements are given in Ref. 4.

In conclusion, the present measurements indicate that when intermittency factor is measured using a circuit of the type described above, calibration by direct photography of the signals may be necessary. It further indicates that even with such calibration there might be errors of the magnitude of $\pm 10\%$ in certain sections of the measured profile.

References

[1] Townsend, A. A., "The Fully Developed Turbulent Wake of a Circular Cylinder," *Australian Journal of Scientific Research*, Vol. 2, 1949, pp. 451–468.

[2] Corrsin, S. and Kistler, A. L., "Free Stream Boundaries of Turbulent Flows," TN 3133, 1954, NACA.

[3] Bradbury, L. J. S., "The Structure of a Self-preserving Turbulent Plane Jet," *Journal of Fluid Mechanics*, Vol. 23, part 1, 1965, pp. 31–64.

[4] Seshagiri, B. V., "Properties of Turbulent Interacting Wakes," Ph.D. thesis, 1971, Dept. of Mechanical Engineering, University of Waterloo, Waterloo, Ontario.

An Invalid Equation in the General Momentum Theory of the Actuator Disk

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THE general momentum theory of the actuator disk model of a propeller,[1] including a helicopter rotor, is incorrect. An unverified equation is included in the theory which, when combined with the other equations, leads to a contradiction. Specifically, the equation of axial momentum for the propeller states that the total thrust is expressible in terms of integrals of the wake parameters. Then the differential form of that equation is applied to the separate annular elements of the propeller. It is this application which leads to a contradiction. In fact, it was noted in Ref. 1, p. 187, that "The validity of this equation has not been established and its adoption may imply the neglect of the mutual interference between the various annular elements . . ."

The actuator disk model is the following. An incompressible, inviscid fluid with a uniform freestream axial velocity $V \geqq 0$, passes through an actuator disk of radius R. The actuator disk is normal to the freestream and is rotating at constant angular velocity Ω. The fluid, in passing through the disk, experiences a discontinuous increase in pressure p' and the angular velocity changes from zero to ω, while the axial component u and radial component v are continuous across the disk. The general nature of the flow behind the disk is shown in Fig. 1.

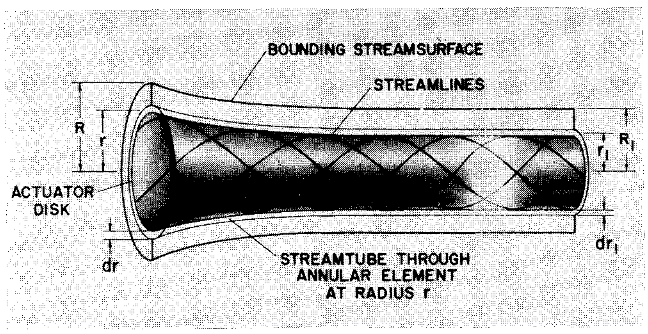

Fig. 1 Flow pattern.

As indicated in Fig. 1, r is the radial distance of any annular element of the propeller disk. In the final wake, let p_1 be the presssure, u_1 the axial velocity and ω_1 the angular velocity at a radial distance r_1 from the axis of the slipstream. p_0 is the pressure outside the slipstream and ρ the density.

Now from the equation of axial momentum for the propeller we have[1]

$$T = \rho \int_0^{R_1} u_1(u_1 - V)\, dS_1 - \int_0^{R_1} (p_0 - p_1)\, dS_1 \tag{1}$$

This equation is generally accepted in the differential form

$$dT = \rho u_1(u_1 - V)\, dS_1 - (p_0 - p_1)\, dS_1 \tag{2}$$

But we shall show that this equation is invalid.

The differential element of thrust at radius r is given by[1]

$$dT = \rho(\Omega - \tfrac{1}{2}\omega)\omega r^2\, dS \tag{3}$$

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Also[1]

$$p_0 - p_1 = \tfrac{1}{2}\rho(u_1{}^2 - V^2) - \rho(\Omega - \tfrac{1}{2}\omega_1)\omega_1 r_1{}^2 \qquad (4)$$

By equating (2) and (3), using Eq. (4) and $dS = 2\pi r\,dr$, $dS_1 = 2\pi r_1\,dr_1$

$$dr/dr_1 = [(u_1 - V)^2 + (2\Omega - \omega_1)\omega_1 r_1{}^2]r_1/(2\Omega - \omega)\omega r^3 \qquad (5)$$

Together with Eq. (5) we have the boundary condition

$$r = 0 \quad \text{at} \quad r_1 = 0 \qquad (6)$$

and from angular momentum conservation

$$\omega_1 r_1{}^2 = \omega r^2 \qquad (7)$$

Hence, we may proceed as follows to determine the functions $\omega_1(r_1)$, $u_1(r_1)$, $r(r_1)$, and $\omega(r)$. Given $\omega_1(r_1)$ then $u_1(r_1)$ is determined by Eq. (4) evaluated at $r_1 = R_1$ for the boundary value and the equation[1]

$$\tfrac{1}{2}d/dr_1\,(u_1{}^2 - V^2) = (\Omega - \omega_1)(d/dr_1)(\omega_1 r_1{}^2) \qquad (8)$$

Next we would like to use the system of Eqs. (5, 6, and 7) to determine $r(r_1)$ and $\omega(r)$. However, Eq. (5) is undefined at $r_1 = 0$. For the sake of the following argument, we assume $R = r(R_1)$ is known. Then we use Eqs. (5) and (7) to integrate backwards over the range from $r_1 = R_1$ to $r_1 = 0$ to obtain the curve $r(r_1)$.

However, we shall now show that for a certain class of functions $\omega_1(r_1)$, which are allowed by the theory, the integral curve $r(r_1)$ of Eq. (5) with $R = r(R_1)$ is bounded from above by a curve which crosses the positive r_1 axis. Hence, Eq. (6) cannot be satisfied and the theory is impossible to satisfy.

Let $\omega_1(r_1)$ be a non-negative function with a continuous derivative on the interval $0 \leqq r_1 \leqq R_1$, and not be zero except possibly at $r_1 = 0$. From Eqs. (4) and (8) we have

$$u_1{}^2 - V^2 = (2\Omega - \omega_1)\omega_1 r_1{}^2 + 2\int_{r_1}^{R_1} \omega_1{}^2 r_1\,dr_1 \qquad (9)$$

Since ω_1 is bounded, say $\omega_1 \leqq M$, the first term on the right side of Eq. (9) approaches zero as $r_1 \to 0$. The second term on the right side of Eq. (9) is positive and increases as $r_1 \to 0$. Hence, there is a point r_0 such that for $r_1 \leqq r_0$

$$(u_1 - V)^2 + (2\Omega - \omega_1)\omega_1 r_1{}^2 \geqq K > 0 \qquad (10)$$

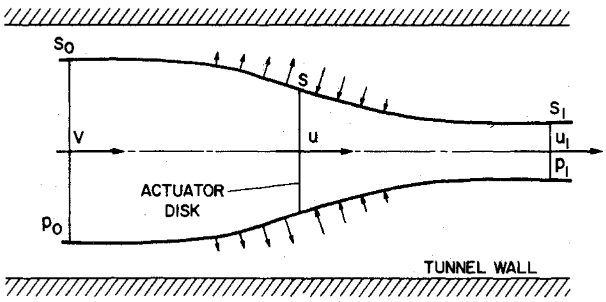

Fig. 2 Disk operating in tunnel.

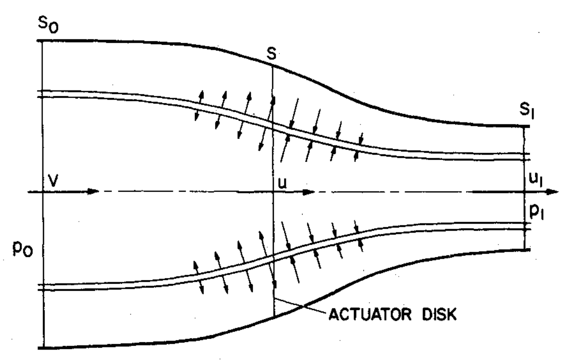

Fig. 3 Forces on streamtube.

Starting at the point $r = R$ and $r_1 = R_1$ we use Eqs. (5) and (7) to integrate into $r_1 = r_0$. For $r_1 \leqq r_0$ we have using Eq. (10), in Eq. (5)

$$dr/dr_1 \geqq Kr_1/(2\Omega - \omega)\omega r^3 \qquad (11)$$

As indicated in Fig. 1, the flow in different annular elements does not mix, i.e., continuity of flow for annular elements.[1] This condition insures that $r(r_1)$ is a one-to-one function, which in turn insures that $2\Omega - \omega$ is not negative for $r_1 \leqq r_0$.

Next, using $\omega > 0$, Eq. (7), and $0 < \omega_1 \leqq M$ in Eq. (11)

$$dr/dr_1 \geqq Kr_1/2\Omega\omega r^3 = Kr_1/2\Omega\omega_1 r_1 r^2 \geqq K/2\Omega M r_1 r \qquad (12)$$

Hence, the integral curve $r(r_1)$ is bounded from above by the integral curve for the equation

$$dr_b/dr_1 = K/2\Omega M r_1 r_b \qquad (13)$$

Equation (13) has the solution

$$r_b{}^2(r_1) = r^2(r_0) - (K/\Omega M)\ln(r_0/r_1) \qquad (14)$$

$r_b{}^2(r_1)$ crosses the positive r_1 axis and bounds from above the curve for $r^2(r_1)$. Hence, Eq. (6) cannot be satisfied.

The case of bounded ω_1 arose while studying the problem of optimum performance[2] when pressure terms are kept in the equations. In that case the optimum flow ω_1 is bounded. As a result of that work, we can give the following physical interpretation. When a pressure gradient is present, the circulation at radius r_1' in the wake affects the thrust produced for $r_1 \leqq r_1'$, i.e., there is "mutual interference between the various annular elements," as speculated by H. Glauert.

Further physical insight can be gained by considering the derivation[1] of Eq. (1). By considering the actuator disk operating in a tunnel (see Fig. 2) we obtain the momentum equation for the flow inside the slipstream.

$$T - X + p_0 S_0 - \int p_1\,dS_1 = \int \rho u_1(u_1 - V)\,dS_1 \qquad (15)$$

X is the resultant axial force of the pressure on the lateral boundary of the column of fluid forming the slipstream and S_0 is the cross section of the slipstream far upstream from the disk. As the cross section of the tunnel becomes very large, X approaches the limiting value

$$X = p_0(S_0 - S_1) \qquad (16)$$

and we obtain Eq. (1).

Now consider an annular element of the actuator disk, see Fig. 3; the momentum equation for the flow inside the streamtube which passes through the annular element is

$$dT - dX + p_0\,dS_0 - p_1\,dS_1 = \rho u_1(u_1 - V)\,dS_1 \qquad (17)$$

dX is the resultant axial force of all the pressure on the inner and outer lateral boundaries of the streamtube. For Eq. (2) to be valid, we would have to prove

$$dX = p_0\,dS_0 - p_0\,dS_1 \qquad (18)$$

Thoma[3] discusses three attempts to derive Eq. (2) with the simplification $p_0 = p_1$. But these attempts fail to prove that the axial force dX on the sides of the streamtube is balanced by the forces $p_0\,dS_0 - p_0\,dS_1$ on the ends. Now in our case, where pressure gradients are taken into consideration, we have shown that Eq. (2) is invalid. Hence, we can conclude that the balance of forces represented by Eq. (18) does not occur.

Actually, the problem of determining the flowfield of an actuator disk model can be properly formulated[4] in terms of finding the solution to a nonlinear elliptic partial differential equation with boundary values.

References

[1] Durand, W. F., ed., *Aerodynamic Theory, Vol. IV*, pp. 182–193.

[2] Wu, J. C., Sigman, R. K., and Goorjian, P. M., "Optimum Performance of Hovering Rotors," CR, NASA.

[3] Thoma, D., "Grundsatzliches zur einfachen Strahltheorie der Schraube," *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, Vol. 16, 1925, pp. 206–208.

[4] Yao-tsu Wu, T., "Flow Through a Heavily Loaded Actuator Disk," *Schiffstechnik*, Vol. 9, No. 47, 1962, pp. 134–138.